Week 11

7 Suppose V is finite-dimensional and $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and every vector in null P is orthogonal to every vector in range P. Prove that there exists a subspace U of V such that $P = P_U$.

Sol^n We claim that $U = \text{range}\, P$ satisfies our condition.

Recall: • If U is a f.d. v. subsp. of V, the orthogonal projection $P_U \in \mathcal{L}(V)$ of V onto U is defined by $P_U v = u$ where $v \in V$ s.t. $v = u + w$ for $u \in U$ $w \in U^\perp$

• Suppose $S \subset V$ is a subset. The orthogonal complement $S^\perp$ of S is defined by

$$S^\perp = \{v \in V : \langle v, s \rangle = 0 \text{ for all } s \in S\}$$

Claim 1: $U^\perp = \ker P$.

Let $w \in \ker P$. From the question $\langle w, u \rangle = 0$ $\forall u \in U$

$\therefore$ $w \in U^\perp$ and $\ker P \subset U^\perp$

By Fund. Thm of Linear Maps, $\dim \ker P + \dim \text{range}\, P = \dim V$

On the other hand $U \oplus U^\perp = V$ by 6.47

$\therefore$ $\dim U^\perp = \dim V - \dim U = \dim \ker P$.

Hence $U^\perp = \ker P$ by Ex 2C Q1

Claim 2 $P = P_U$

Let $v \in V$ $\exists u \in U$ $w \in U^\perp$ s.t. $v = u + w$. Since $U = \text{range}\, P$ $\exists u' \in V$ s.t. $Pu' = u$. Therefore

$$Pv = Pu + Pw = P^2 u' + 0 = Pu' = u = P_U v$$

Since this is true for all $v \in V$, $P = P_U$.

Midterm 2

Q1 True or False (Only false statements are shown).

(c) Let V be a finite dimensional vector space. For any diagonalizable $S, T \in \mathcal{L}(V)$, $S + T \in \mathcal{L}(V)$ is also diagonalizable.

TRUE **FALSE**

(d) Let T be a linear operator on a vector space V. Then the set of eigenvectors corresponding to an eigenvalue of T is a subspace of V.

TRUE **FALSE**

(e) For any linear operator T on $\mathbb{R}^7$, there exists an ordered basis β of $\mathbb{R}^7$ such that $\mathcal{M}(T, \beta)$ is upper triangular.

TRUE **FALSE**

(h) Let V be a real inner product space and $v, w \in V$. Then $\|v + w\| = \|v\| + \|w\|$ if and only if there exists a real number c such that $v = cw$ or $w = cv$.

TRUE **FALSE**

c) Consider $V = \mathbb{R}^2$ $T(x,y) = (x+y, 0)$ $S(x,y) = (0,y)$

T, S have eigen basis $((1,0),(1,-1))$ and $((1,0),(0,1))$

$\therefore$ diagonalizable

However $(T+S)(x,y) = (x+y, y)$ If $(T+S)(x,y) = \lambda(x,y)$

$\lambda(x,y) = (x+y, y)$ $\therefore \lambda x = x+y$ $\lambda y = y$

$(\lambda-1)x = y$ $(\lambda-1)y = 0$ $\therefore \lambda = 1$ or $y = 0$

If $y=0$ then $\lambda = 1$ or $x = 0$ Since we assume $(x,y) \neq (0,0)$

$\lambda = 1$ and $y = 0$ $E(\lambda, T+S) = \{(x,0) : x \in \mathbb{R}\}$ is only 1 dim'l, $\therefore$ not diag.

d) Since 0 is not an eigenvector, the set of all e. vectors does not contain it and cannot be a subsp.

e) Define $T(x_1, x_2, \cdots, x_7) = (-x_2, x_1, x_3, \cdots, x_7)$

It is a lin. op. on $\mathbb{R}^7$.

(A shorter proof) Consider $\beta = (e_1, \cdots, e_7)$ Then $\mathcal{M}(T,\beta) = \begin{bmatrix} \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & 0 \\ 0 & I_5 \end{bmatrix}$

Its char. poly. is $(x^2+1)(x-1)^5$ which does not split over $\mathbb{R}$. If there exist an upper triangular matrix repr., then it should split.

(A longer proof) Assume that $\beta = (f_1, \cdots, f_7)$ is a basis of $\mathbb{R}^7$ s.t. $\mathcal{M}(T,\beta)$ is upper triangular.

Then we have $Tf_i \in \operatorname{Span}(f_1, \cdots, f_i)$ for $i = 1, \cdots, 7$

Let $V_i = \operatorname{Span}(f_1, \cdots, f_i)$ for $i = 0, \cdots, 7$ $(V_0 = \{0\})$ $U = \operatorname{Span}(e_1, e_2)$

Claim If W is a T-invar subsp of V s.t. $W \cap U \neq \{0\}$ then $U \subset W$.

Pf Pick $0 \neq v \in W \cap U$ $v = c_1e_1 + c_2e_2$ for some $c_1, c_2 \in \mathbb{R}$ not both zero

$Tv = -c_2e_1 + c_1e_2 \in W$ Note that $\frac{c_1}{c_1^2+c_2^2} v - \frac{c_2}{c_1^2+c_2^2} Tv = e_1 \in W$

and $\frac{c_2}{c_1^2+c_2^2} v + \frac{c_1}{c_1^2+c_2^2} Tv = e_2 \in W$ $\therefore$ $U = \operatorname{Span}(e_1, e_2) \subset W$ $\square$

Suppose m is the smallest +ve int. s.t. $V_m \cap U \neq \{0\}$.

It exists since $V_7 = \mathbb{R}^7 \supset U$.

By def. of m $V_{m-1} \cap U = \{0\}$ (Take $V_0 = \{0\}$) By Claim $U \subset V_m$

$\therefore$ $V_m \supset V_{m-1} \oplus U$ However $m = \dim V_m \geq \dim V_{m-1} + \dim U = m-1+2 = m+1$

Contradiction. Therefore no such m. $\therefore$ No such β.

h) Take $V = \mathbb{R}$ with standard inner product $\langle x, y \rangle = xy$. Take $v = 1$ $w = -1$

Then $v = -1 \cdot w$ but $\|v+w\| = |1-1| = 0$ while $\|v\| + \|w\| = 1 + 1 = 2 \neq 0$.

3. (9 pts) Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of all real polynomials of degree at most 2 and $\beta = \{1, x, x^2\}$ be an ordered basis of $\mathcal{P}_2(\mathbb{R})$. Define a linear operator T on $\mathcal{P}_2(\mathbb{R})$ by

$$T(p(x)) = xp'(x) - p(1).$$

(a) Find the matrix $\mathcal{M}(T, \beta)$;

(b) Find all the eigenvalues of T;

(c) Determine if T is diagonalizable. If so, find an eigenbasis α of T and the corresponding matrix $\mathcal{M}(T, \alpha)$.

a) $T(1) = x \cdot 0 - 1 = -1$ $\quad T(x) = x \cdot 1 - 1 = x - 1$

$T(x^2) = x \cdot 2x - 1 = 2x^2 - 1$

$\therefore \mathcal{M}(T, \beta) = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

b) Since $\mathcal{M}(T,\beta)$ is upper-triangular its eigenvalues are entries on the diagonal.

$\therefore$ The eigenvalues of T are $-1, 1, 2$

c) Since T has $3 = \dim P_2(\mathbb{R})$ distinct e.value T is diagonalizable

Solve $\overbrace{(\mathcal{M}(T,\beta) - \lambda I)}^{A_\lambda} v = 0$ for $\lambda = -1, 1, 2$

$\lambda = -1$ $\quad \begin{bmatrix} 0 & -1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ Pick $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda = 1$ $\quad \begin{bmatrix} -2 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Pick $v_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

$\lambda = 2$ $\quad \begin{bmatrix} -3 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Pick $v_3 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$

$\therefore \alpha = (1, 1-2x, 1-3x^2)$ is an eigen basis of T and

$\mathcal{M}(T, \alpha) = \begin{bmatrix} -1 & & \\ & 1 & \\ & & 2 \end{bmatrix}$.